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Fourier Transforms from a Weighted Trace Map

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Abstract—The class of Generalised Hadamard Transforms includes the Fourier, Generalised, Discrete Fourier, Walsh-Hadamard, Complex Hadamard and Reverse Jacket Transforms. The Generalised Hadamard Transforms may be partly classified by signal length, by group of entries in the transform matrix and by a recently introduced third parameter, the jacket width of the transform matrix. Here we introduce a weighted trace map, which realises the Fourier Transform as an exponential weighted sum of Galois Ring traces. We give examples of Fourier Transforms with jacket width 0, jacket width 1 and maximum jacket width (half the signal length). We show the Fourier Transforms of length 4^k with entries in $\{\pm 1, \pm i\}$ obtained using the weighted trace map from the Galois ring $GR(4, k)$ have jacket width 2^{k-1} .

I. INTRODUCTION

There are many discrete signal transforms whose transform matrices have entries on the unit circle. For instance, the Discrete Fourier Transform (DFT) and the Walsh-Hadamard Transform (WHT) are both special cases of the Fourier Transform (FT) found by interpreting the Cooley-Tukey Fast Fourier Transform in terms of abelian group characters [12]. The family of discrete Generalised Transforms $\{(GT)_r, 0 \leq r \leq a-1\}$ for signals of length 2^a [3, 10.2] includes the WHT as case $r = 0$, the complex BIFORE transform as case $r = 1$ and the DFT of order 2^a as case $r = a-1$. Both the WHT and DFT are suboptimal discrete orthogonal transforms, but each has wide application for signal and image representation and processing, and efficient, easily implemented fast algorithms exist to compute them. The WHT is well-known for its applications to image compression, Walsh representation of data sequences in image coding, and for signal separation in CDMA mobile communication [7].

The transform matrices of all these transforms fall into the class of Butson Hadamard (BH) matrices [2].

For multi-phase or multilevel signals where entries outside the complex n^{th} roots of unity are needed, the first author generalised the class of BH matrices to define a Generalised Hadamard Transform (GHT) [6]. The GHT includes all the above transforms and other families, such as the unimodular complex Hadamard transforms, whose matrices have entries on the complex unit circle, and the Reverse Jacket Transforms of [10]. The former are important in high-energy and theoretical physics and play a critical role in the theory of quantum information [17]. The latter are so-called because the unitary matrix representing the transform has a border (“jacket”) and centre which switch some elements under inversion. This

family includes the weighted Hadamard Transforms [11], with a border of $\{\pm 1\}$ and real centre entries of absolute value greater than 1, which weight the mid-band frequencies of the signal more.

In [6], those GHT matrices which had a border of $\{\pm 1\}$ were called jacket matrices. The maximum width of a border of $\{\pm 1\}$ in a generalised BH matrix is proposed here as a new parameter for classifying transforms. In particular it can be applied to Fourier Transforms, and ranges from 0 or 1 for DFTs, depending whether the signal length is odd or even, to the maximum possible for WHTs.

In Section II, a description of GHT matrices and their properties is summarised from [6]. Each normalised GHT matrix has an associated jacket width $w \geq 0$. (We have modified the definition of [6] slightly.) This is the maximum width of a complete border of rows and columns consisting entirely of ± 1 s in the normalised GHT matrix, up to permutations of the rows and columns.

In Section III we restrict to GHT with entries which are roots-of-unity, so the GHT matrix is a BH matrix. We introduce a weighted trace map which represents the Fourier Transform (FT) as an exponential weighted sum of Galois Ring trace maps. Examples of jacket transforms of varying widths and signal alphabets are derived. In Section IV we summarise our results and pose further problems.

II. GENERALISED HADAMARD TRANSFORMS

In this most general situation, we work with a signal alphabet from a ring R with unity 1, and a transform matrix with entries from its group of units R^* . This includes signal alphabets from the real field $\mathbb{R}$, the complex field $\mathbb{C}$ and the Galois Fields $GF(p^e)$. It also includes Galois Ring alphabets $GR(p^e, k)$, in which case the group of units is $\{\sum_{j=0}^{e-1} p^j u_j : u_j \in \mathcal{T}, u_0 \neq 0\}$, where $\mathcal{T}$ is the Teichmüller set [13]. If, however, we need to distinguish signal values x and $-x$, the ring R must have characteristic $\neq 2$. The transform matrix is a Generalised Butson Hadamard (GBH) matrix.

Definition 2.1: Suppose R is a ring with unity 1, group of units R^* and that $\text{char } R$ does not divide v . A square matrix M of order $v \geq 2$, with entries from a subgroup $N \leq R^*$, is a *Generalised Butson Hadamard (GBH) matrix*, if

$$MM^* = M^*M = vI_v,$$

where M^* is the transpose of the matrix of inverse elements of M : $m_{ij}^* = (m_{ji})^{-1}$. It is denoted $GBH(N, v)$, or $GBH(|N|, v)$ if N is finite.

If $\mathbf{x}$ is a signal of length v from R , where $v \in R^*$, and M is a $GBH(N, v)$, a *Generalised Hadamard Transform* (GHT) of $\mathbf{x}$ is

$$\hat{\mathbf{x}} = M \mathbf{x} \quad (1)$$

and an *Inverse Generalised Hadamard Transform* (IGHT) of $\hat{\mathbf{x}}$ is

$$\mathbf{x} = v^{-1} M^* \hat{\mathbf{x}}. \quad (2)$$

A GBH matrix is always equivalent to a *normalised* GBH matrix, which has first row and column consisting of all 1s. By taking the inner product of any non-initial row of a normalised GBH matrix M with the all-1s first column of M^* , we see that the sum of the entries in any row of M , apart from the first, must equal 0, and similarly for rows of M^* (columns of the matrix of inverses $M^{(-1)} = [m_{ij}^{-1}]$). The tensor product of two GBH matrices over the same group N is a GBH matrix over N .

Throughout this section, let C be an indexing set of order v (sometimes C is an abelian group such as the group $\mathbb{Z}_v$ of integers mod v or the direct product $\mathbb{Z}_2^e$ of e copies of $\mathbb{Z}_2$, but C may be non-abelian or the group structure may be irrelevant). Let $D_n \leq \mathbb{C}^*$ denote the group of complex n^{th} roots of unity.

Definition 2.2: Let R be a ring with unity 1. A normalised $GBH(N, v)$ matrix K indexed by $C = \{1, \dots, v\}$ with entries from $N \leq R^*$ is a *jacket matrix* if it is permutation equivalent to a matrix of the form

$$\tilde{K} = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & * & \dots & * & \pm 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & * & \dots & * & \pm 1 \\ 1 & \pm 1 & \dots & \pm 1 & \pm 1 \end{bmatrix} \quad (3)$$

where the central entries $*$ are from N . The *jacket width* of K is $w \geq 1$ if K is permutation equivalent to a jacket matrix $\tilde{K}$ in which rows $1, \dots, w, v - w + 1, \dots, v$ and columns $1, \dots, w, v - w + 1, \dots, v$ all consist of ± 1 and w is maximal for this property. If K is *not* permutation equivalent to any jacket matrix it has *jacket width* 0.

Since all non-initial ± 1 rows and columns sum to 0 in R , the order v of a jacket matrix K must be even.

Corollary 2.3: If v is odd, any normalised $GBH(N, v)$ matrix has jacket width 0.

The family of Discrete Fourier Transform (DFT) matrices of order v

$$\mathcal{F}_v = [\omega^{jk}]_{0 \leq j, k \leq v-1}, \quad (4)$$

where $\omega = e^{-2\pi i/v}$, provides examples of jacket width 1 in every even order.

Example 2.4: The DFT matrix $\mathcal{F}_{2n}$ of order $2n$ is permutation equivalent to a jacket matrix of width 1 with entries in D_{2n} .

Proof. By [10, Theorem 1, Definition 1] $\mathcal{F}_{2n}$ is permutation equivalent to a jacket matrix of width ≥ 1 . But for a fixed j , $\omega^{jk} = \pm 1$ for all $0 \leq k \leq 2n - 1$ if and only if $\omega^j = \pm 1$ if and only if either $j = 0$ (the initial row and column) or $j = n$. Thus the width is exactly 1. $\square$

At the other extreme, the WHT matrix S_t of order 2^t

$$S_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad S_t = \bigotimes^t S_1, \quad t \geq 2, \quad (5)$$

with jacket width 2^{t-1} , is “all jacket”. In fact this holds for any Hadamard matrix transform.

Example 2.5: A jacket matrix K of order $2n$ has *maximum* width n if and only K is a normalised Hadamard matrix.

This means a maximal width jacket matrix of order $2n$ has entries in $D_2 = \{\pm 1\}$ and either it is S_1 or n is even.

In [6, Lemma 3.5] it is shown that if K_i is a width $w_i \geq 1$ jacket matrix with entries from R^* , for $i = 1, 2$, then their tensor product $K_1 \otimes K_2$ is a jacket matrix of width at least $2w_1w_2$. A similar argument shows that if $w_i = 0$ then $K_1 \otimes K_2$ has width at least w_j ($j \neq i$).

Theorem 2.6: If K_i is a normalised $GBH(N, v_i)$ matrix of jacket width w_i with entries from R^* , for $i = 1, 2$, then $K_1 \otimes K_2$ is a jacket matrix of width at least $2w_1w_2$, if $w_1w_2 \geq 1$, and has jacket width at least w_j ($j \neq i$), if $w_i = 0$.

Consequently we are interested in finding the fundamental jacket matrices, which are not permutation equivalent to any such tensor products.

Definition 2.7: [6, Definition 4.1] A jacket matrix of order $2n$ is a *primary* jacket matrix K_n if it is minimal with respect to tensor product, that is, there are no jacket matrices K, K' such that K_n is permutation equivalent to $K \otimes K'$.

Clearly any jacket matrix of width 1, such as the even length DFTs of Example 2.4, is primary, but by Example 2.5 this sufficient condition is not necessary for a jacket matrix to be primary.

Corollary 2.8: A jacket matrix of width 1 is primary. A jacket matrix of order $2n \geq 4$ and maximum width $n = 2k$ is primary whenever k is odd.

The tensor product of two jacket matrices is a jacket matrix, but by Theorem 2.6, for a tensor product of normalised GBH

matrices to be jacket matrix it is enough that one factor is a jacket matrix.

Corollary 2.9: [6, Theorem 5.1] Let B be a normalised GBH matrix and K a jacket matrix, both with entries in R^* . Then $B \otimes K$ is a jacket matrix.

The width w of a GHT matrix is a third parameter, after v and N , we can use to construct and classify signal transforms over varying signal alphabets.

III. CONSTRUCTION OF ROOT-OF-UNITY TRANSFORMS AND THE WEIGHTED TRACE MAP

In this section we concentrate on transforms whose entries are all complex m^{th} roots of unity, so $N = D_m$. The transform matrices are all BH matrices $GBH(m, v)$. From now on, let the index set C be an abelian group of order v and exponent m . (The exponent $m \geq 0$ of a group G which is written multiplicatively with identity 1, is the smallest integer such that $g^m = 1$ for every $g \in G$, so, for example, $\mathbb{Z}_v$ has exponent v and $\mathbb{Z}_2^e$ has exponent 2).

The BH matrix (4) of the DFT for signals of length v uses as indexing set the cyclic group $C = \mathbb{Z}_v$ of exponent v . The construction generalises for any finite abelian group C of order v to a Fourier Transform (FT) matrix [12].

An irreducible character of a finite abelian group C of order v and exponent m is any group homomorphism from C to the multiplicative group $D_m = \langle e^{2i\pi/m} \rangle \subset \mathbb{C}$ of all complex m^{th} roots of unity. The character group $\widehat{C} = \text{Hom}(C, D_m)$ of all irreducible characters of C is isomorphic to C .

Definition 3.1: Let C be a finite abelian group of order v and exponent m and fix an ordering $C = \{c_1, \dots, c_v\}$. Fix an isomorphism $\chi : C \rightarrow \widehat{C}$ and denote the image of $c \in C$ by χ_c . The Fourier Transform (FT) of a complex-valued function $\varphi : C \rightarrow \mathbb{C}$ is the function $\widehat{\varphi} : C \rightarrow \mathbb{C}$ given by

$$\widehat{\varphi}(c_k) = \sum_{\ell=1}^v \varphi(c_\ell) \chi_{c_k}(c_\ell), \quad 1 \leq k \leq v$$

and the Inverse Fourier Transform (IFT) of $\widehat{\varphi}$ is

$$\varphi(c_\ell) = v^{-1} \sum_{k=1}^v \widehat{\varphi}(c_k) \overline{\chi_{c_k}(c_\ell)}, \quad 1 \leq \ell \leq v,$$

where $\bar{z}$ denotes the complex conjugate of z . The FT matrix, with entries in D_m ,

$$\mathcal{F}_C = [\chi_{c_k}(c_\ell)]_{1 \leq k, \ell \leq v} \quad (6)$$

is a $GBH(D_m, v)$.

When $C = \mathbb{Z}_v$, the FT is the usual DFT of (4). When $C = \mathbb{Z}_2^a$, the FT is the WHT of (5), since any homomorphism is represented by a linear Boolean function $L_{\mathbf{u}} : \mathbb{Z}_2^a \rightarrow \mathbb{Z}_2$ with $L_{\mathbf{u}}(\mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle$, where $\langle \mathbf{u}, \mathbf{v} \rangle$ is the inner product of

vectors $\mathbf{u}$ and $\mathbf{v}$ over $GF(2)$, and vice versa.

Any two abelian groups C, C' of the same order v but different exponents m, m' , respectively, will determine Fourier Transforms with entries in distinct roots-of-unity subgroups $D_m, D_{m'}$, respectively. For each v , there are at most finitely many such transforms. The simplest illustration is given for $v = 4$ by $C = \mathbb{Z}_4$ of exponent 4 (the DFT) and $C' = \mathbb{Z}_2^2$ of exponent 2 (the WHT).

More subtly, any two non-isomorphic abelian groups of the same order v and exponent m , will determine different Fourier Transforms. Again, there are only finitely many such transforms. A small example for $v = 16$ and $m = 4$ is given by $C = \mathbb{Z}_4 \times \mathbb{Z}_4$ and $C' = \mathbb{Z}_4 \times \mathbb{Z}_2^2$.

Thirdly, an abelian group C of order v and exponent m may have several distinct formulations as a direct product. This is the basis of the Cooley-Tukey FFT and is exemplified by $C = \mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.

Finally, for each C , the precise form of the FT will depend on the choice of isomorphism $\chi : C \rightarrow \widehat{C}$ in (6).

In the general case, we write C as a direct product. If $C \not\cong \mathbb{Z}_{p^e}$ for any prime power p^e , then C has at least one decomposition as a direct sum of proper subgroups. Write $C = C_1 \oplus \dots \oplus C_n \cong C_1 \times \dots \times C_n$, using the standard isomorphism between a finite internal direct sum and external direct product. There may be several ways of doing this.

For example, by the fundamental theorem for finite abelian groups, we may assume, if required, that $C_j = \mathbb{Z}_{m_j}$ is cyclic and $m_j | m$, where m is the exponent of C . In this case, the isomorphism $\chi : C \rightarrow \widehat{C}$ may be selected as follows [7]. Write $x_j = (0, \dots, 0, 1, 0, \dots, 0)$ for the standard generator of the j^{th} direct summand of C and select $\omega_j = \omega^{m/m_j} = e^{2i\pi/m_j}$ as the m_j^{th} root of unity used to define the character group $\widehat{C_j}$. Then, for all $\mathbf{c} = (c_1, c_2, \dots, c_n)$, $\mathbf{b} = (b_1, b_2, \dots, b_n) \in C$,

$$\begin{aligned} \chi_{\mathbf{c}}(\mathbf{b}) &= \chi_{\mathbf{c}}\left(\sum_{j=1}^n b_j x_j\right) = \prod_{j=1}^n (\chi_{\mathbf{c}} \downarrow C_j)(b_j x_j) \\ &= \prod_{j=1}^n \chi_{\mathbf{c}_j}(b_j) = \prod_{j=1}^n \omega_j^{c_j b_j} \end{aligned}$$

by [9, Chapter 9], where the character $(\chi_{\mathbf{c}} \downarrow C_j)$ of the subgroup C_j is the restriction of $\chi_{\mathbf{c}}$ to C_j . That is, for all $\mathbf{c} = (c_1, c_2, \dots, c_n)$, $\mathbf{b} = (b_1, b_2, \dots, b_n) \in C$,

$$\chi_{\mathbf{c}}(\mathbf{b}) = \omega^{\mathbf{c} * \mathbf{b}}, \quad \text{where} \quad \mathbf{c} * \mathbf{b} = \sum_{j=1}^n c_j b_j m / m_j. \quad (7)$$

In particular, when $m_j = m$ for all $1 \leq j \leq n$, $\mathbf{c} * \mathbf{b} = \mathbf{c} \cdot \mathbf{b} = \sum_{j=1}^n c_j b_j$.

Now we present a new formulation of the isomorphism $\chi : C \rightarrow \widehat{C}$ in terms of Galois Ring traces. We no longer require all the C_j to be cyclic. Instead, assume that, for each j , $C_j =$

$(\mathbb{Z}_{m_j})^{k_j}$, where $m_j = p_j^{e_j}$ is a prime power. The primes p_j are not necessarily distinct. In this case, each character can be realised in terms of trace maps for Galois Rings. The Galois Ring $GR(p^e, k)$ is a degree k Galois extension of the ring $\mathbb{Z}_{p^e}$. See McDonald [13] for example, for more details on Galois Rings, or Hammons et al [4] for a brief outline in the case $p^e = 4$.

The trace map for the Galois Ring $GR(p^e, k)$ is defined as follows. Using the p -adic representation $u = \sum_{j=0}^{e-1} p^j u_j$ of elements of $GR(p^e, k)$, the Frobenius automorphism $f : GR(p^e, k) \rightarrow GR(p^e, k)$ is

$$f(u) = \sum_{i=0}^{e-1} p^i u_i^p.$$

When $e = 1$, $GR(p, k)$ is the Galois Field $GF(p^k)$ and f is the usual Frobenius automorphism for $GF(p^k)$. The trace map $T_{(p^e, k)} : GR(p^e, k) \rightarrow \mathbb{Z}_{p^e}$ over $GR(p^e, k)$ is defined by

$$T_{(p^e, k)}(u) = u + f(u) + f^2(u) + \dots + f^{k-1}(u). \quad (8)$$

It is easy to see that the trace map satisfies the following properties: for any $u, v \in GR(p^e, k)$ and $\alpha \in \mathbb{Z}_{p^e}$,

- 1) $T_{(p^e, k)}(u + v) = T_{(p^e, k)}(u) + T_{(p^e, k)}(v)$.
- 2) $T_{(p^e, k)}(\alpha u) = \alpha T_{(p^e, k)}(u)$.
- 3) $T_{(p^e, k)}$ is surjective.

As well, $T_{(p^e, k)}$ is equi-distributed. Note that $u \in GR(p^e, k)$ is invertible if and only if $u_0 \neq 0$. Thus every non-invertible element of $GR(p^e, k)$ can be written uniquely as $u = \sum_{i=j}^{e-1} p^i u_i$ for a smallest $j = 1, 2, \dots, e-1$ for which $u_j \neq 0$, so we can represent elements of $GR(p^e, k)$ by $u^{(j)} = \sum_{i=j}^{e-1} p^i u_i$, $j = 0, 1, 2, \dots, e-1$.

Theorem 3.2: [16, Lemma 2.1] Let $B_j = \{p^j t \mid t = 0, 1, 2, \dots, p^{e-j} - 1\}$, $j = 0, 1, 2, \dots, e-1$. For each $j = 0, 1, 2, \dots, e-1$, as x ranges over $GR(p^e, k)$, $T_{(p^e, k)}(xu^{(j)})$ takes elements in B_j equally often; that is, $p^{e(k-1)+j}$ times.

Importantly, since the additive group of $GR(p^e, k)$ is isomorphic to $(\mathbb{Z}_{p^e})^k$, of order p^{ek} and exponent p^e , we know by 1) above that each additive character of $(\mathbb{Z}_{p^e})^k$ may be realised in terms of the corresponding trace map:

$$\chi_u(v) = \omega^{T_{(p^e, k)}(uv)}, \quad u, v \in GR(p^e, k), \quad \omega = e^{2\pi i/p^e}. \quad (9)$$

Now we can use (7) and (9) to realise each character of C as a weighted trace map, written as a weighted sum of Galois Ring traces. This map first appears in [14], for the case $k_1 = k_2 = \dots = k_n$, where it is called the trace-like map.

Definition 3.3: For $1 \leq j \leq n$ let p_j be prime such that $p_1 \leq p_2 \leq \dots \leq p_n$. Let $m_j = p_j^{e_j}$ and let $C_j = (\mathbb{Z}_{m_j})^{k_j}$. For each $1 \leq j \leq n$, let $T_{(m_j, k_j)} : GR(m_j, k_j) \cong C_j \rightarrow \mathbb{Z}_{m_j}$ be the corresponding trace map of (8).

Let $C = C_1 \times \dots \times C_n$ have exponent m and let $\omega = e^{2i\pi/m}$. Define the *weighted trace map* $T : C \rightarrow \mathbb{Z}_m$ by

$$T(\mathbf{b}) = \sum_{j=1}^n T_{(m_j, k_j)}(b_j) m/m_j, \quad \mathbf{b} \in C. \quad (10)$$

Therefore, for each fixed $\mathbf{c} \in C$,

$$T(\mathbf{c}\mathbf{b}) = \sum_{j=1}^n T_{(m_j, k_j)}(c_j b_j) m/m_j, \quad \mathbf{b} \in C \quad (11)$$

is a distinct homomorphism from C to $\mathbb{Z}_m$.

Theorem 3.4: Let C and T be as in Definition 3.3. For each $\mathbf{c} \in C$, the isomorphism $\chi : C \rightarrow \hat{C}$ may be selected as

$$\chi_{\mathbf{c}}(\mathbf{b}) = \omega^{T(\mathbf{c}\mathbf{b})}, \quad \mathbf{b} \in C. \quad (12)$$

The corresponding FT matrix, with entries in D_m , is

$$\mathcal{F}_C = [\omega^{T(\mathbf{c}\mathbf{b})}]_{\mathbf{c}, \mathbf{b} \in C}. \quad (13)$$

These Fourier Transform matrices (13) are all instances of a class of matrices called *cocyclic*, investigated by the authors and colleagues over the past decade [5], [8], [14].

We finish by determining the jacket widths of several infinite families (indexed by signal length) of root-of-unity transforms. The first two follow immediately from Corollary 2.3 and Theorem 2.6.

Example 3.5: Let C be an abelian group of odd order v and exponent m . Then the Fourier Transform matrix $\mathcal{F}_C$ has jacket width 0. The DFT of (4) is the case $C = \mathbb{Z}_v$.

Example 3.6: For any $\mathcal{F}_C$ of jacket width 0 in Example 3.5, the tensor product $\mathcal{S}_1 \otimes \mathcal{F}_C$ has jacket width 1. The index group is $\mathbb{Z}_2 \times C$, of order $2v$ and exponent $2m$. If $C = \mathbb{Z}_v$ then $\mathbb{Z}_2 \times C \cong \mathbb{Z}_{2v}$ and $\mathcal{S}_1 \otimes \mathcal{F}_C$ can be permuted to the DFT matrix $\mathcal{F}_{2v}$ of (4).

The last example gives a family of Fourier Transforms with entries in $\{\pm 1, \pm i\}$, of orders 4^k , $k \geq 1$ and jacket width 2^{k-1} .

Lemma 3.7: Let $C = C_1 = (\mathbb{Z}_4)^k$, of order 4^k and exponent 4, so $T = T_{(2^2, k)}$ in Definition 3.3. Then $\mathcal{F}_C$ is a jacket matrix of width 2^{k-1} , with entries in $\{\pm 1, \pm i\}$.

Proof. If c is not invertible then $T_{(2^2, k)}(cb)$ takes values 0 and 2 equally often, by Theorem 3.2. Hence $\chi_c(b)$ will be a row consisting only of ± 1 .

By the 2-adic representation, $c = 2t$ where $t \in \mathcal{T}$, the Teichmüller set and $|\mathcal{T}| = 2^k$. $\square$

The jacket width and other properties of an arbitrary Fourier Transform matrix (13) is currently under investigation.

IV. SUMMARY AND FURTHER WORK

We have shown that after signal length v and entry alphabet group N , a third parameter, jacket width $w \geq 0$, may be used to classify Generalised Hadamard Transforms. For real signals, the real-valued centre-weighted Hadamard Transforms, with jacket width at least 1, weights the mid-band frequencies of the signal more than the WHT does.

For complex signals of length v we have shown that for any abelian group C of order v and exponent m there is a Fourier Transform $[(e^{2i\pi/m})^{T(\mathbf{cb})}]_{\mathbf{c}, \mathbf{b} \in C}$ defined by a weighted trace map $T : C \rightarrow \mathbb{Z}_m$, which is a weighted sum of Galois Ring traces. Since abelian groups of order v can have different exponents, or the same exponent but different decompositions as direct products, in most cases there is a choice of Fourier Transforms.

The following problems could next be considered.

What properties of a signal over complex-valued alphabets $N = D_m$ or $N = \mathbb{C}^*$, say, does the jacket width of a Generalised Hadamard Transform measure?

There are infinite families (indexed by order $2n$) of primary jacket matrices with minimum jacket width 1 ($N = \mathbb{Z}_{2n}$) and with maximum jacket width $n = 2k$, k odd ($N = \{\pm 1\}$). Given $v = 2n$, do primary jacket matrices exist for some N in all possible jacket widths $w = 1, 2, \dots, n$?

Jacket width is defined for normalised $GBH(N, v)$ up to permutation equivalence. If all equivalence operations on a $GBH(N, v)$ are allowed, i.e. left multiplication of a row or right multiplication of a column by an element of N , could the jacket width change? In other words, is jacket width an invariant of each equivalence class of $GBH(N, v)$?

Fourier Transforms admit fast decompositions. For a given v , what is the optimal FFT obtained from the various possible FT of Theorem 3.4?

Finally, how do the GHTs above relate to the algebraic classification of Fourier-type signal transforms in [15]?

REFERENCES

- [1] K. T. Arasu and W. de Launey, Two-dimensional perfect quaternary arrays, *IEEE Trans. Inform. Theory* 47(4) (2001) 1482–1493.
- [2] A. T. Butson, Generalised Hadamard matrices, *Proc. Amer. Math. Soc.* 13 (1962) 894–898.
- [3] D. F. Elliott and K. R. Rao, *Fast Transforms: Algorithms, Analyses, Applications*, Academic Press, New York, 1982.
- [4] A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane and P. Solé, The $\mathbb{Z}_4$ -linearity of Kerdock, Preparata, Goethals, and related codes, *IEEE Trans. Inform. Theory* 40 (1994) 301–319.
- [5] K. J. Horadam, An introduction to cocyclic generalised Hadamard matrices, *Discrete Appl. Math.* 102 (2000) 115–131.
- [6] K. J. Horadam, A generalised Hadamard Transform, Proc. 2005 IEEE International Symposium on Information Theory, 4–9 September 2005, Adelaide, Australia, 1006–1008.
- [7] K. J. Horadam, *Hadamard Matrices and Their Applications*, PUP, Princeton, 2006, to appear.
- [8] K. J. Horadam and P. Udaya, Cocyclic Hadamard codes, *IEEE Trans. Inform. Theory* 46 (4) (2000) 1545–1550.
- [9] G. James and M. Liebeck, *Representations and Characters of Groups*, C.U.P., Cambridge, 1993.
- [10] M. H. Lee, B. Sunder Rajan and J. Y. Park, A generalized reverse jacket transform, *IEEE Trans. Circuits Syst. II*, 48 (7) (2001) 684–690.
- [11] S.-R. Lee and M. H. Lee, On the reverse jacket matrix for weighted Hadamard Transform, *IEEE Trans. Circuits Syst. II*, 45 (3) (1998) 436–441.
- [12] D. K. Maslen and D. N. Rockmore, Generalised FFTs - a survey of some recent results, *DIMACS Ser. Discr. Math. Theoret. Comp. Sci.* 28 (1997) 183–237.
- [13] B. R. McDonald, *Finite rings with identity*, Marcel Dekker, New York, 1974.
- [14] N. Pinnawala and A. Rao, Cocyclic Codes over $\mathbb{Z}_n$, Res. Rept. 2005/03, Math. Dept., RMIT University, 2005.
- [15] M. Puschel and J. M. F. Moura, The algebraic structure in signal processing: time and space, preprint 2005.
- [16] A. Rao and N. Pinnawala, New linear codes over $\mathbb{Z}_{p^s}$ via the trace map, Proc. 2005 IEEE International Symposium on Information Theory, 4–9 September 2005, Adelaide, Australia, 124–126.
- [17] W. Tadej and K. Zyczkowski, A concise guide to complex Hadamard matrices, preprint 2005, arXiv:quant-ph/0512154v1, 19 Dec 2005.