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Comment on "Relative variance of the mean squared pressure in multimode media: rehabilitating former approaches" [J. Acoust. Soc. Am. 136, 2621-2629 (2014)]

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Running title: The variance of the transmission function

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Models for the statistics of responses in finite reverberant structures, and in particular for the variance of the mean square pressure in reverberation rooms, have been studied for decades. It is therefore surprising that a recent JASA communication has claimed that the literature has gotten the simplest of such calculations very wrong. Monsef *et al.* ["Relative variance of the mean squared pressure in multimode media: rehabilitating former approaches" J. Acoust. Soc. Am. 136, 2621-2629 (2014)] have derived a modal-based expression for the relative variance that differs significantly from expressions that have been accepted since 1969. This comment points out that the Monsef formula is clearly incorrect, and then for the interested reader, points out the subtle place where they made their mistake.

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I. INTRODUCTION

The theory for power variances in reverberant structures and rooms has attracted attention for many decades now. It is clear that at large modal overlap $M=2\pi n\gamma$, where n is the modal density (modes per unit angular frequency ω) and γ is the dissipation rate (units of nepers per time) of acoustic amplitude, the underlying modal character ought not be discernible and (time-domain) responses ought be indistinguishable from real Gaussian random processes under an exponential decay envelope. Thus at large overlap the relative variance is unity, i.e. the expected fluctuations in the measured power are comparable to the power itself. These are called Ericson fluctuations in nuclear physics (Legrand *et al.*, 1995).

At more modest overlap the variance is expected to be much larger, and the search for an accurate theory has attracted considerable attention. A variance larger than unity at small M is easily understood if one recalls that, at small M , the resonances are relatively distinct. In this case room response varies strongly as the frequency of excitation coincides, or not, with an eigenfrequency of the room. Early theory (Lyon, 1969) based on an assumption of uncorrelated eigenfrequencies suggested the relative variance should be $1+K^2/M$ for all M , with K related to the statistics of the mode shapes, equal to $\langle u^4 \rangle / \langle u^2 \rangle^2$, the ratio of the mean fourth power of a mode amplitude to the square of the mean square.

Davy (1981; 1986; 1987), and Lyon (1969), showed that this formula is modified if the eigenfrequencies are correlated, predicting at large M , $1+(K^2-3/2)/M$ for what Lyon called nearest-neighbor (NN) eigenfrequency repulsion. Weaver (1989a) pointed out that those NN eigenfrequency correlations lack the long-range rigidity predicted by random matrix theory (Brody *et al.*, 1981). He then showed that the full eigenfrequency correlations derived from the Gaussian Orthogonal Ensemble (GOE) of Random Matrix Theory (RMT) modifies the formula

further. Like others Weaver obtained $1+K^2/M$ at small M (regardless of correlations). But at large M he found that long range spectral rigidity leads to the expression $1+(K^2-3)/M$.

These theoretical predictions are further modified if the modal decay rates γ are taken to vary randomly between the modes ("decay curvature") (Burkhardt and Weaver, 1996; Lobkis *et al.*, 2000).

The value of K has also received attention. Conventional RMT predicts that the mode shapes are real Gaussian random processes. To the extent that actual modes are complex, K ought to be less than three. Langley and Brown (2004a; 2004b) and Langley and Cotoni (2005) have argued that K ought be less than three based on realizing that the modes of finite sized orthogonal random matrices will not be Gaussian. Fits of measured variances to theory have suggested values of K between 2.5 and 4 (Langley and Brown, 2004a; b; Langley and Cotoni, 2005; Jacobsen, 2012).

These formulas are readily generalized for the variances and covariances of measurements of power averaged over multiple sources and/or receivers at different working frequencies (Davy, 1981; 1986; 1987).

A distinct theory that does not consider modes but proceeds directly from the random orthogonal dissipative Hamiltonian and includes the effects of decay curvature and complex modes has been suggested also, but it is difficult to calculate with (Rozhkov *et al.*, 2004).

Differences over realistic choices for K , the importance of decay curvature, the range of level rigidity and focus on covariances and/or multiple sources and receivers notwithstanding, all theorists seem to be in agreement that the relative variance for power transmission between distant points ought be exactly (for the special and simple case of no eigenfrequency correlations and no decay curvature and real modes and a single source and receiver) $1+K^2/M$. It is therefore

surprising that in a recent communication, Monsef *et al.* (2014) have suggested that the correct expression for these conditions is $1+(K^2-2)/M$. Either Monsef *et al.* (2014) have discerned an error that has escaped the attention of others for fifty years, or they have made a mistake. The mystery is compounded further as they, realizing their unusual claim requires corroboration, show numerical Monte Carlo calculations that appear to be in agreement with their formula.

This comment proceeds by first demonstrating that their expression violates a required mathematical inequality and therefore cannot be correct. It then identifies where in their analysis they made their error. It then speculates as to what aspects of their Monte Carlo calculation led to its agreement with the erroneous theory.

II. RELATIVE VARIANCE OF A MODAL RESPONSE SUM

Monsef *et al.*'s analytic derivation starts as do others with an expression equivalent to

$$p(\mathbf{x}, \omega) = \sum_m \frac{u_m(\mathbf{x})u_m(\mathbf{s})}{\omega^2 - \omega_m^2 - 2i\omega\gamma_m} \quad (1)$$

This describes the complex steady state pressure at a position $\mathbf{x}$ due to a harmonic point source at position $\mathbf{s}$. The u are the normal modes (assumed real in the current application). The ω_m are the natural frequencies and γ_m are the modes' absorption rates. We seek the statistics, in particular the mean square of $|p|$ and the mean fourth power of $|p|$. To estimate these one must make assumptions about the statistics of the u and of the ω_m .

The mode shapes u_n and u_m are traditionally assumed uncorrelated for $n \neq m$, and the positions $\mathbf{x}$ and $\mathbf{s}$ are assumed such that $u_m(\mathbf{x})$ is uncorrelated with $u_m(\mathbf{s})$. It is convenient to take the u_n to be centered real Gaussian random numbers. A key quantity, it transpires, is the ratio of the mean fourth power of u to the square of the mean square: $K = \langle u^4 \rangle / \langle u^2 \rangle^2$. If the u are Gaussian, this number is three. For the purposes of a thought experiment, one may assume any $K \geq 1$.

One must also consider the correlation, if any, amongst the ω_m . It is widely argued that realistic reverberant structures have the natural frequency correlations of the GOE (Weaver, 1989a; b; Davy, 1990; 2009). Monsef *et al.* take in their theory the older and simpler assumption that the natural frequencies are uncorrelated. They implicitly also take the spectral density of eigenfrequencies n to be constant, and all modes to have identical absorption rates γ . While those choices are arguably unphysical, they remain acceptable for the purposes of a thought experiment or for Monte Carlo simulations.

These assumptions about the u and the natural frequencies may be – and often ought to be – relaxed for application to real reverberant structures, but the present purposes do not require it.

Having described the statistics of the constituents u and ω_m and γ_m in the expression for p , it should be straightforward to then calculate the mean square of $|p|$ and the mean fourth power of $|p|$. The quantity of particular interest is the relative variance

$$\text{Rel var} = \frac{\langle (|p(\mathbf{x}, \omega)|^2 - \langle |p(\mathbf{x}, \omega)|^2 \rangle)^2 \rangle}{\langle |p(\mathbf{x}, \omega)|^2 \rangle^2} = \frac{\langle |p(\mathbf{x}, \omega)|^4 \rangle}{\langle |p(\mathbf{x}, \omega)|^2 \rangle^2} - 1 \quad (2)$$

which by construction is positive semi-definite. Theory in the literature has long agreed that, under the condition that the natural frequencies are uncorrelated, this quantity is $1+K^2/M$ where M is the modal overlap which equals $2\pi n\gamma$. Monsef *et al.* calculate that Relvar is $1+(K^2-2)/M$. That this result is impossible is readily recognized; Monsef *et al.*'s expression can take negative values for admissible values of K and M . For example, consider the case that all $u^2 = 1$. Then $K=1$ and $\text{Relvar}=1-1/M$ is negative for small M .

That others obtain similar looking but approximate expressions like $1+(K^2-3)/M$ for relative variance in the presence of natural frequency correlations is not in violation of the requirement that Relvar be nonnegative, because such expressions are derived under an assumption of high overlap M and are not intended to apply at small M .

III. A KEY QUANTITY FOR THE THEORY OF Relvar

It behooves us to determine how Monsef *et al.* made their error. The error is subtle and occurs in their Appendix where they evaluate the expectation

$$E_{|\psi|^2|\psi|^2} = \left\langle \sum_{m \neq k} \frac{1}{|\omega - \omega_m - i\gamma|^2 |\omega - \omega_k - i\gamma|^2} \right\rangle \quad (3)$$

If the eigenfrequencies are uncorrelated (as assumed here and purportedly by Monsef *et al.*) the two factors under the summation are uncorrelated.

$$E_{|\psi|^2|\psi|^2} = \sum_{m \neq k} \left\langle \frac{1}{|\omega - \omega_m - i\gamma|^2} \right\rangle \left\langle \frac{1}{|\omega - \omega_k - i\gamma|^2} \right\rangle \quad (4)$$

The expectation becomes (Davy, 1981)

$$E_{|\psi|^2|\psi|^2} = J(J-1) \left\langle \frac{1}{\Omega} \int \frac{d\omega_m}{|\omega - \omega_m - i\gamma|^2} \right\rangle^2 = \left(1 - \frac{1}{J}\right) \frac{n^2 \pi}{\gamma} \quad (5)$$

where J is the number of modes and Ω is the frequency range (such that modal density $n=J/\Omega$). On recognizing that $1/J$ is negligible one obtains the standard result and, after a bit more algebra, the result Relvar= $1+K^2/M$. It is noteworthy that to derive this one must take each of the J eigenfrequencies to range uniformly and independently in the ensemble over the full spectrum Ω .

Monsef *et al* proceed differently. In their average of equation (4) over the ensemble of possible values for ω_m and ω_k , they take each eigenfrequency to vary randomly and independently over a range (of width $\Delta\omega$ which must be less than a quantity of order γ in order to justify the approximation used later to obtain their equation (A4) from their equation (A3)) centered on mean values that are indicated here by overbars, $\overline{\omega_m}$ and $\overline{\omega_k}$ respectively. In that case their evaluation of the sum in equation (4) is dominated by those few k and m such that $\overline{\omega_m}$ and $\overline{\omega_k}$ are near the working frequency ω . Thus their effective J is not large, but rather of order

$M = 2\pi n\gamma$. In lieu of Davy's $1-1/J$, they obtain $1-1/M$; the consequences for Relvar are then as they describe.

So which is the correct form for the distribution of an eigenfrequency? Davy distributes ω_m over a long range $\Omega \gg 1/n$. Monsef *et al.* distribute it over a narrow range around some mean. If the modes are ordered conventionally, by increasing value, such that ω_m is the m^{th} lowest eigenfrequency, then as Monsef *et al.* imply, the m^{th} eigenfrequency will have some mean value $\bar{\omega}_m$ (of order m/n), and be distributed over some limited range around that mean. If one takes the eigenfrequencies to be independently distributed and then ordered, the standard deviation of ω_m will be of order $\Delta\omega = m^{1/2}/n$, and not necessarily smaller than γ as Monsef *et al.* required. Moreover, if the eigenfrequencies are chosen independently and then ordered there will be further correlations: the m^{th} mode has precisely $m-1$ modes with lower frequency. (In Monsef's model the correlations are a bit weaker, than that.) In consequence, Monsef *et al.*'s decomposition of the sum in equation (3) to derive equation (4) becomes invalid. It may be possible to employ conventional ordering for calculations of Relvar, but the resulting correlations between ω_k and ω_m would make the calculation difficult. It is much simpler to eschew conventional ordering, as it serves no useful purpose. Davy's formulation in equation (5) is the correct one, with $1/J$ negligible. It is also much simpler.

Monsef's assumed distribution of ω_m corresponds to high spectral rigidity. Their spectrum is highly uniform; a range $\delta\omega$ of their spectrum has an expected number of eigenfrequencies $n\delta\omega$, with a fixed variance that their model takes to be a constant proportional to the square of their quantity $\Delta\omega$. Poisson distributions have number variances (Brody *et al.*, 1981) that grow linearly with range $\delta\omega$. The GOE has a number variance that grows logarithmically with range

$\ln(\delta\omega)$ (Brody *et al.*, 1981). Inasmuch as Monsef *et al.*'s variance is even less than the GOE's at long range, one may characterize their spectrum as having high spectral rigidity. As is well known, GOE rigidity leads to a prediction, at large enough M , that Relvar is $1 + (K^2 - 3)/M$, so it is not surprising that Monsef *et al.*'s rigidity might lead to something similar.

IV. NUMERICAL RESULTS

Monsef *et al.* admit that many readers might be skeptical that their expression $1 + (K^2 - 2)/M$ could be correct, because of the long established $1 + K^2/M$. In order to remove doubts, they present the results of a Monte Carlo (MC) simulation of equation (1) with Gaussian random real u (so that $K=3$) and purportedly uncorrelated natural frequencies. Their MC results appear to be in agreement with $1 + (K^2 - 2)/M$. It is not obvious why it agrees, but it is probable that their MC ensemble of natural frequencies contained a good deal of spectral rigidity. They appear to consider a base distribution of natural frequencies that is a "picket fence" in which all frequencies are separated by the same amount $1/n$. They then perturb it randomly by smearing the distribution of each natural frequency over a finite range (taken to be $2\pi/n$). This is consistent with our description above of their theory. If so, one can see why their MC numerical results corresponded so well to $1 + (K^2 - 2)/M$ and not $1 + K^2/M$. Their MC natural frequencies were in fact well correlated, with significant rigidity. Rigidity for the GOE case implies Relvar is $1 + (K^2 - 3)/M$ at large M (Weaver, 1989a), i.e. $1 + 6/M$ for Gaussian mode shapes with $K=3$. That this is not very different from their theory $1 + 7/M$ suggests that their spectral rigidity is responsible for the numerical agreement. In short, their MC spectrum was not Poissonian.

V. OTHER ERRORS

There are two other points unrelated to the discrepancy between $1 + K^2/M$ and $1 + (K^2 - 2)/M$, that ought to be mentioned.

Monsef *et al.* make a claim about the approximation introduced by Weaver (1989a) in which the effect of rigidity was – at least for large enough M – simply and analytically accounted for by taking the Y_2 function equal to a Dirac delta (Monsef *et al.*'s equation (8)). They claim that doing so is equivalent to the case of uncorrelated natural frequencies with a caveat that no pair of frequencies can be identical. That is an incorrect characterization; it would be correct only if Y_2 was replaced by a Kronecker delta.

Monsef *et al.* claim that their integral (25) converges, even if one inserts the usual three-dimensional modal density (their equation (24)). This is incorrect. Their $|\psi|^2$ diminishes asymptotically inversely with the square of the frequency f (not the fourth power like they claim); the modal density increases with the square of the frequency. Their integrand is therefore asymptotically independent of frequency and their integral does not converge. The error here is merely one of exposition; $|\psi|^2$ would diminish like f^{-4} if they had not already imposed the otherwise convenient approximation that the working frequency is of the same order as the relevant eigenfrequencies. Their integral (25) would converge using an unapproximated ψ , but not for the approximate ψ as written in their equation (13). On replacing $n(\omega_n)$ with a constant $n(\omega_w)$ (as do most others) their integral again converges even with the approximate ψ . The net error is negligible in practice.

VI. CONCLUSIONS

Our familiar and long-established and accepted expressions for relative variance when there is no decay curvature and no spectral rigidity need not be re-examined. The canonical work by Lyon and Davy does not need rehabilitation. This does not however mean that the question of power transmission variances is settled. The effects of decay curvature, of complex modes and

finite GOE correlation range, with the consequent uncertainty as to the best choice for K remain unclear.

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